Dynamic Epistemic Logic, Game Theory, and Cognition

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1 Abstract

This paper explores the interface between logic and game theory including dynamic epistemic logic and game logic as well as how these topics interact with cognitive science.

keywords: game theory, epistemic logic, cognitive science

2 Introduction

Logic stands at the core of many fields. From philosophy to computer science to economics to everyday problems, the ability to reason and perform logic has set humans apart from the rest of the animal kingdom. There are many different kinds of logic to represent many different forms of reasoning. This paper will be focusing on epistemic logic. More specifically, dynamic epistemic logic. Epistemic logic is a kind of modal logic that lets us reason about knowledge and belief. Although it traces its early roots to the 1950s, it wasn't until Jaakoo Hintikka's 1962 book *Knowledge and Belief* that the groundwork for epistemic logic was laid. Following Hintikka's work, dynamic epistemic logic was created by combining epistemic logic with dynamic logic. Dynamic epistemic logic expands epistemic logic to deal with multi-agents that change their knowledge after an event occurs. Using dynamic epistemic logic, we can critically examine problems in game theory. Game theory is the branch of mathematics and economics that analyzes strategies in competitive games. Often times, the outcomes of each participant also depends on the choices of the other participants in the game. This paper explores the ways dynamic epistemic logic and game theory interplay and how these two topics affect the field of experimental cognitive science. The first section of this paper will focus on the syntax and semantics of epistemic logic while the second section will show how dynamic epistemic logic differs from classic epistemic knowledge. The third section will introduce games and how they can be models for dynamic epistemic knowledge. Finally, the fourth section will try to connect these two subjects to cognition and social interaction.

3 Epistemic Logic

Epistemic logic is under the branch of modal logic. Unlike classical logic, modal logic does not rely on strict true/false assignments. It's able to express modality and therefore we can reason things like possibility and necessity about time, knowledge, and beliefs. Like classical logic, epistemic logic has it's own syntax and semantics that govern how it is applied.

3.1 Syntax

In the language of epistemic logic or L_{el}^{C}, $AGTS = \{1, 2, 3...n\}$ is the finite set of agents while $PROP$ is the set of all the propositional letters. In addition, there are three main modal operators. The K operator is the knowledge operator. If $K_j\phi$, then the agent j knows ϕ. There is also the common knowledge operator C_a and the distributive knowledge operator D_a. ϕ is common knowledge if and only if everyone knows ϕ and everyone knows that everyone knows ϕ and everyone knows that everyone knows that everyone knows ϕ and so on. On the other hand, distributed knowledge is when the agents have pieces of information that if pooled together, they could conclude ϕ. L_{el}^{C}'s grammar can be defined as:

L_{el}^{C}: $\phi ::= p| - \phi|\big(\phi \wedge \phi\big)|K_j\phi|C_A\phi|D_A\phi$

where $p \in PROP$, $j \in AGTS$, and $A \subseteq AGTS$

3.2 Semantics

Since epistemic logic is a modal logic, it relies on Kripke semantics. To express possibility and necessity, Kripke semantics uses possible worlds and interpretation of truth within possible worlds to model modality. In the epistemic model $M = \big(W, R_1, ...R_n, I\big)$, the set W is a non-empty set of possible worlds, I is an interpretation or valuation operation, and R_n is the accessibility relation between propositions and worlds. Each world is given a set of true or false propositions. A world that is said to be accessible to any agent j is one that does not have any contradictions to the current world j is in. We get $\big(a, b\big), \in R_j$ if world b is compatible with agent j's information about world a. For two worlds to be compatible, it must be the case that

$ER\big(i\big)\big(j\big)\big(s\big) = (\big(\forall p\big)\big(\forall q\big)\Big(\Big(\big(K_s p_i\big) \wedge q_j\Big) \rightarrow \neg\big(p \rightarrow -q\big) \vee -\big(q \rightarrow -p\big)\Big)$

where s is the subject, i, j are worlds, and p, q are propositional statements

4 Dynamic Epistemic Logic

While epistemic logic lets us reason about the modalities of knowledge, dynamic epistemic logic or DEL adds a richer level of computation. By adding dynamic logic to epistemic logic, DEL allows us to reason about a knowledge and how it changes after a certain event occurs. There are three different models in a DEL system of reasoning. An epistemic model lets you express beliefs prior to a certain situation. An event model lets you represent beliefs about an event while that event is happening. Finally, a product update model shows you how agents have changed their beliefs either during or after an event. There are many different kinds of events that can provide information that can change an agent's belief. However, the simplest and most common one is a public announcement

4.1 Public Announcement Logic

A public announcement is a new piece of information that is told to all the agents. Depending on what this information is, the players can change their beliefs and continue to play after this event. Public announcement has it's own branch of logic called public announcement logic or PAL or L_{PAL}. It's grammar can be defined as such:

$L_{PAL} : \phi ::= p| - \phi|\big(\phi \wedge \phi\big)|K_j\phi|\big[\psi!\big]\phi$

where $j \in AGTS$

Notice that $L_{PAL} and L_{EL}$ have very similar syntax. The only new operator is $\big[\psi!\big]\phi$ which is called the dynamic action modality. The truth condition for this operator is as follows:

$M, w \models \big[\psi!]\phi$ iff if $M, w \models \psi$ then $M^\psi, w \models \phi$

where $M^\psi := \big(W^\psi, R_1^\psi, ...R_n^\psi, I^\psi\big)$

and $W^\psi := \big[w \in W; M, w \models \psi\big]$

$R_k^\psi := R_j \cap \big(W^\psi \times W^\psi\big)$ for all $j \in \big(1, 2, ...n\big)$ and

$I^\psi := I(w)$ for all $w \in W^\psi$

Intuitively, the dynamic action modality $\big[\psi!\big]\phi$ means that after some true announcement ψ, ϕ still holds. It also means that any knowledge that includes $-\psi$ has to be removed in order to prevent contradictions of beliefs. This system of filtering out contradicting statements is called information update.

4.1.1 Muddle Children Problem

One of the most famous cases of public announcement logic is the muddy children puzzle. Two children,let's say Annie and Billy, are playing outside. Their father calls them in but before he lets them inside the house, he informs them that at least one of the children has mud on their forehead. Annie can see Billy's forehead but not her own and the same is true for Billy. Let's assume that both of the children are dirty. We can use PAL to model their knowledge.

Let s be the case that Annie (A) is dirty and t be the case that Billy (B) is dirty.

The initial situation is that both Annie and Billy are dirty

$Q, r \models s \wedge t$

However, neither of them know that they are dirty. So,

$Q, r \models \big(- K_A s \wedge -K_A - s\big) \wedge \big(- K_B t \wedge -K_B - t\big)$

Then after the public announcement from their father that at least one of them is dirty, both of them know that at least one of them is dirty

$Q, r \models [s \vee t!]\Big(K_A\big(s \vee t\big)\Big) \wedge \Big(K_B\big(s \vee t\big)\Big)$

But, they still do not know which one of them is the dirty one:

$Q, r \models [s \vee t!]\big(- K_A s \wedge -K_A - s\big) \wedge \big(- K_B t \wedge -K_B - t\big)$

However, the father tells them again that at least one of them are dirty and since they can't figure out who, they reason it must be both of them.

$Q, r \models [s \vee t!]\big[\big(- K_A s \wedge -K_A - s\big) \wedge \big(- K_B t \wedge -K_B - t\big)!\big]\big(K_A s \wedge K_B t\big)$

5 Games and DEL

One of the best application dynamic epistemic logic has is to game theory. With representations of knowledge and belief of multiple rational agents that interact, it's not hard to see why dynamic epistemic logic is an interest to game theorists. There are many different kinds of games like cooperative vs non cooperative, discrete or continuous, sequential vs simultaneous. For the purpose of this paper, however, we'll be focusing on perfect information vs imperfect information games. In a perfect information game, agents are aware of the rules of the game, who the players are, and what strategies and actions they are using. In an imperfect game however, that last part goes away. Players are uncertain of the opponent's move and have to think of strategies that take that to account.

5.1 Perfect Information Games

Models of perfect information games run like process graphs or Kripke models. Therefore, we only need to use modal logic to accurately represent them. Although Kripke's semantics use possible worlds, in game theory, we use possible states. For any model M of a game tree,

$\mathbf{M} = \big(S, \big[R_a | a \in A\big], V\big)$

where S is the set of states, R_a are the relations of possible action or moves of type a for agent A, and V is the valuation of propositional letter that denote the properties of the states.

There are also operations like $turn_i$,end, and win_i that denote when a it's a player's turn to move, when the game has ended, and if they have won. And, the string $< a > p$ intuitively means action a leads to outcome p.

5.1.1 Dynamic Modal Logic

We can also use dynamic modal logic to talk about perfect information games. Using basic actions like $a, b...$ and operations like $(\cup)$ for choice, (;) for sequential execution,(*) for finite iteration, and (ϕ)? as a test to see if assertion ϕ holds, we can describe more intricate values than modal logic. There are also quantifiers in this language that say

$< A > \phi$ holds in some A-successor and $\big[A\big]\phi$ holds in all A-successors

So, using this language we can translate any perfect information game tree into dynamic modal logic. For example to describe a winning strategy for agent i in a zero-sum game, it can be written like this:

$WIN_i \leftrightarrow \big(end \wedge win_i\big) \vee \big(turn_i \wedge < A > win_i\big) \vee (-turn_i \wedge [A]win_i)$

5.2 Imperfect Information Games

What's more interesting for dynamic epistemic logic is imperfect information games. Unlike perfect information games, what and how much you know about the imperfect information game can affect the outcome of the entire game. But, unlike perfect information games, there is a lot of uncertainty in imperfect games. To account for this, the grammar for imperfect logic games is this:

$\mathbf{M} = \big(S, \big[R_a | a \in A\big], \big[\sim_i | i \in I\big], V\big)$

where $\sim_i$ is agent i's uncertainty.

An agent can be uncertain about a lot of things in an imperfect game. Depending on the game, she may not know where she is in the game, what she just played, what the other person played, if it's her turn, or if the game is still going.

Along with uncertainty, iteration is another crucial part in epistemic logic for games. Players can know about the knowledge and information, it can give them an advantage. To notate this we write

$\big(K_i K_j \phi\big) \wedge \big(K_i - K_j K_i K_j \phi\big)$

This says that subject i knows that subject j knows ϕ and agnet i knows that j does not know that i knows j knows ϕ, giving agent i a big advantage over j.

6 Cognition

6.1 Cognition and Logic

Unlike game theory and logic, there isn't a distinct connection between cognitive science and dynamic epistemic logic. Even though cognitive science focuses on the brain and it's processes, there's a social part to cognition that can be modeled into DEL. The easiest and most used cognitive interaction is language. Although we often talk to ourselves in our heads, language is essential for communication and social functioning. Every time we talk to someone outside of ourselves, we are making an interaction. This interaction, whether it's cooperative or competitive, can be understood through game theory and therefore, can be understood through logic. The clearest way we can see DEL in language is through questions and answers.

Say you don't know which way the supermarket is. You find a friendly stranger to ask if the road you are on now leads to the supermarket. The stranger answers yes and you head down the road to indeed find the market. This interaction can be modeled in dynamic epistemic logic using the PAL method mentioned in section 4.

Assume p is the supermarket is on the same road you, agent A, are on

At the initial state, you did not know whether you were on the right road or not

$N, s \models \big(- K_A(p \vee -p)\big)$

Then, you ask the stranger if you were on the right road in which they told you the statement yes making the location of the supermarket common knowledge. You perform information update really quickly and come up with this result:

$N, s \models ([p!](K_A p))$

Although this is a very simple and rudimentary modeling, it highlights the ability of dynamic epidemic logic to parse something so human into a logical language.

6.2 Cognition and Game Theory

We've seen how cognition fits into logic but how does it help game theory? Although memory, rationality, and motivation fit into game theory, one of the biggest ways higher order cognition helps game theory is belief revision. Belief revision is a less extensive way of information update. During information update, you get rid of every belief that is inconsistent with some new information. With belief revision, you are not throwing away knowledge but tweaking it to be consistent. For example, when playing a game with a new adversary, you may think of them to have certain qualities like aggressive or selfish and play the game according to these beliefs about the other person. However, if your opponent plays fair and even is pleasant to you during the course of the game, you will revise your belief of the other agent and choose to change your own behavior, strategy, and action by playing more fair and becoming pleasant yourself.

7 Conclusion

Logic has a hand in every academic pie. We see it's applications in computer science, in economics, in psychology, and in philosophy. In this paper we covered just two of it's many applications: game theory and cognitive science. Dynamic epistemic logic helps us model a game theoretic strategies and actions into modal logic. It can also help us turn a simple human interaction into a well-formed logical formula. We also explored a little bit of how game theory and cognitive science interact with each other. Although there is extensive research in the connections between game theory and logic as well as game theory and cognitive science, there has been little exploration from logicians into the world of cognitive science. Hopefully future research dives into deeper levels of social interaction and the computational complexity needed in different sorts of interactions. Whatever the result, however, the search is always the best part.

8 References

Baltag, A. and Renne, B. (2018). Dynamic Epistemic Logic. [online] Plato.stanford.edu. Available at: https://plato.stanford.edu/entries/dynamic-epistemic/ [Accessed 7 Dec. 2018].

En.wikipedia.org. (n.d.). Dynamic epistemic logic. [online] Available at: https://en.wikipedia.org/wiki/Dynamicepistemiclogic [Accessed 7 Dec. 2018].

Plato.stanford.edu. (2018). Epistemic Foundations of Game Theory (Stanford Encyclopedia of Philosophy). [online] Available at: https://plato.stanford.edu/entries/epistemic-game/ [Accessed 7 Dec. 2018].

van Benthem, J. (2006). Cognition As Interaction. [online] Available at: http://www.illc.uva.nl/Research/Publications/Reports/PP-2005-10.text.pdf [Accessed 7 Dec. 2018].

van Benthem, J., Pacuit, E. and Roy, O. (2010). Toward a Theory of Play: A Logical Perspective on Games and Interaction. Games.

van Benthem, J. (n.d.). Games in Dynamic-Epistemic Logic. Bulletin of Economics Research, 53(4), pp.219-248.